Langevin equations with multiplicative noise: The mathematics, physics, and thermodynamics of interpretation.

Igor M. Sokolov

Humboldt-Universität zu Berlin, Institut für Physik, Newtonstr. 15, 12161 Berlin, Germany

Stochastic differential equations are quite different from customary differential equations encountered in the description of physical dynamical systems. The right-hand side of an overdamped Langevin equation with a multiplicative white Gaussian noise does not satisfy the assumptions of the existence / uniqueness (Picard-Lindelöf) theorem. The solutions to such equations can only be uniquely defined provided additional restrictions on the kind of solution (the correct interpretation of the stochastic integral) are given. Thus, similarly looking Langevin equations give rise to different Fokker-Planck ones, and, respectively, to different probability density functions of displacements at given times. This freedom of interpretation has a deep rooting in physics, since similar Langevin equations may appear as coarse-grained, and sometimes locally homogenized, descriptions of quite diverse physical situations [1]. The freedom given by the interpretation corresponds to the possibility to incorporate the information lost when passing from the micro- to the mesoscopic level of description (implying a specific kind of a limiting transition) into the ensuing mesoscopic-level picture. The corresponding physical information may come from thermodynamics, from local symmetries of the system, or from experiments, but the overall situation is much more faceted than typically assumed. In the present talk we discuss several aspects of the problem. We start from the history of the topic, and discuss the nature of limiting transitions and limit theorems leading to Langevin equations. We discuss the freedom left for the solution of the Langevin equation after the corresponding limiting transitions are done, and consider models leading to different interpretations of the Langevin equations after the corresponding limits are taken. We consider the thermodynamics of these models, and discuss the connections between their micro-, macro- and mesoscopic levels of description. We moreover consider the intimate connection between the interpretation and the geometric symmetries of transition rates in the corresponding system: if these symmetries are known, they define the interpretation uniquely. The situation can however be more complicated since the local symmetries may be broken. We moreover discuss how the correct interpretation can be obtained from the single-particle tracking experiments [2,3].

[1] I.M. Sokolov, Chemical Physics 375 (2010) 359-363

[2] A. Pacheco-Pozo, M. Balcerek, A. Wyłomanska, K.Burnecki, I.M. Sokolov, and D. Krapf, Phys. Rev. Lett. 133 (2024) 067102

[3] A. Pacheco-Pozo, I.M. Sokolov, R. Metzler, and D. Krapf, New J. Phys. 27 (2025) 064602