The life cycle of random walks on random regular graphs

Ofer Biham, Ido Tishby, and Eytan Katzav

The Hebrew University, Racah Institute of Physics, Jerusalem, Israel

We present analytical results for the trajectories of random walks (RWs) on random regular graphs (RRGs), which consist of N nodes of degree c. Starting from a random initial node i, at each time step an RW hops into a random neighbor of its previous node. Here we focus on landmark events in the life cycle of RWs on RRGs. In particular, we calculate the distribution of times at which these events take place.

First Hitting (FH) Time: In some of the time steps the RW may hop into a yet-unvisited node while in other time steps it may revisit a node that has already been visited before. The first time at which the RW enters a node that has already been visited before is called the first hitting time. The first hitting event may take place either by backtracking to the previous node or by retracing, namely stepping into a node which has been visited two or more time steps earlier. We calculate the distribution $P(T_{\mathrm{FH}} = t)$ of first hitting times, which turns out to be a product of a geometric distribution and a Rayleigh distribution. In the dilute network limit the first hitting process is dominated by backtracking and the mean first hitting time is $\langle T_{\mathrm{FH}} \rangle \sim c$. In the dense network limit it is dominated by retracing and the mean first hitting time is $\langle T_{\mathrm{FH}} \rangle \sim \sqrt{N}$.

First Return (FR) time: We calculate the distribution $P(T_{\mathrm{FR}} = t)$ of first return times of the RW to the initial node i. We distinguish between the first return trajectories in which the RW retrocedes its own steps backwards all the way back to the initial node i and the trajectories in which the RW returns to i via a path that does not retrocede its own steps (and thus must include at least one cycle). In the retroceding scenario the first return time follows an exponential distribution whose mean is of order 1, while in the non-retroceding scenario it follows an exponential distribution whose mean is of order N.

Cover (C) time: The cover time T_{C} is the number of time steps required for the RW to visit every single node in the network at least once. We derive a master equation for the distribution $P_t(S = s)$ of the number of distinct nodes s visited by an RW up to time t and solve it analytically. Inserting $s = N$ we obtain the cumulative distribution of cover times, namely the probability $P(T_{\mathrm{C}} \le t) = P_t(S = N)$ that up to time t an RW will visit all the N nodes in the network. Taking the large network limit, we show that $P(T_{\mathrm{C}} \le t)$ converges to a Gumbel distribution, whose mean is $\langle T_{\mathrm{C}} \rangle \sim N \ln N$.

[1] I. Tishby, O. Biham and E. Katzav, J. Phys. A: Math. Theor. 54, 145002 (2021).

[2] I. Tishby, O. Biham and E. Katzav, J. Phys. A: Math. Theor. 54, 325001 (2021).

[3] I. Tishby, O. Biham and E. Katzav, J. Phys. A: Math. Theor. 55, 015003 (2022).