Hyperbolic band theory

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The notions of Bloch wave, crystal momentum, and energy bands are commonly regarded as unique features of crystalline materials with commutative translation symmetries. Motivated by the recent realization of hyperbolic lattices in circuit quantum electrodynamics, we exploit ideas from algebraic geometry to construct the first hyperbolic generalization of Bloch theory, despite the absence of commutative translation symmetries. For a quantum particle propagating in a large class of hyperbolic lattice potentials, we construct a continuous family of eigenstates that acquire Bloch-like phase factors under a discrete but noncommutative group of hyperbolic translations, the Fuchsian group of the lattice. A hyperbolic analog of crystal momentum arises as the set of Aharonov-Bohm phases threading the noncontractible cycles of a higher-genus Riemann surface that is naturally associated with this group. This crystal momentum lives in a higher-dimensional Brillouin zone torus, known in algebraic geometry as the Jacobian of the Riemann surface, and over which a discrete set of continuous energy bands can be computed. Familiar concepts such as particle-wave (Fourier) duality, point-group symmetries, and the tight-binding approximation are likewise generalized to hyperbolic lattices. We demonstrate our theory by explicitly computing hyperbolic Bloch wavefunctions and bandstructures numerically for hyperbolic lattice potentials associated with a particular Riemann surface of genus two, the Bolza surface.